Pergamon

Int. J. Solids Structures Vol. 34, No. 30, pp. 3923–3939, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(97)00011–5

HIGH FREQUENCY VIBRATIONS AND WAVE PROPAGATION IN ELASTIC SHELLS: VARIATIONAL-ASYMPTOTIC APPROACH

LE KHANH CHAU
Lehrstuhl für Allgemeine Mechanik, Ruhr-Universität Bochum, 44780 Bochum, Germany

(*Received* 30 *May* 1996; *in revised form* 3 *December* 1996)

Abstract—Two-dimensional equations are derived for high-frequency vibrations of linear elastic shells. The derivation is based on the variational asymptotic analysis of the three-dimensional action functional. This guarantees the exactness of the derived equations for the classical and thickness branches of vibrations in the long-wave range. A best short-wave extrapolation is chosen so as to reach the qualitative agreement with the three-dimensional theory in the short-wave range. Comparisons are made with the similar results of the three-dimensional theory. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The classical two-dimensional equations of motion of elastic plates and shells can be used to describe their vibrations in the low-frequency long-wave range (Berdichevsky, 1979). The exact solutions of the three dimensional equations of elasticity for infinite plates, by Rayleigh (1889) and Lamb (1917), confirm this conclusion. A similar situation exists with regard to the classical one-dimensional equations of motion of elastic rods (Pochhammer, 1876). Numerical analysis of Rayleigh-Lamb's and Pochhammer's dispersion equations (Onoe, 1955; Tolstoy and Usdin, 1957) shows that, as the frequency increases, many new branches of the dispersion curves arise. These branches are connected to each other in the complex wave-number plane, signifying the complicated interaction between waves of different branches near the free edge of the plate or rod. As the wave number and the frequency increase, the velocities in the three-dimensional theory have upper limits for all branches, in contrast to the classical two-dimensional theory. Hence, the latter cannot be expected to give good results for the frequencies of modes of vibration of high order.

Timoshenko (1921) was the first who included the effect of transverse shear deformation to derive a one-dimensional theory of flexural motions of bars which gives more satisfactory results for short waves and high modes of vibrations. But Timoshenko's theory and its generalization for plates and shells (Reissner, 1947, statics only; Berdichevsky, 1979) have the shortcoming that they cannot describe satisfactorily the cut-off frequency (corresponding to the zero wave number) and the long-wave asymptotes of the first branch of thickness vibrations.

It was Mindlin (1951) and Mindlin and Medick (1959), who succeeded in deriving two-dimensional equations of motions of plates which gives satisfactory results for dispersion curves of both low-frequency and thickness branches. In their pioneering papers the following method of derivation has been proposed. The displacements are expressed by the expansions in the series of Legendre polynomials of the thickness coordinate. These series expressions are then substituted into the three-dimensional action functional followed by an integration over the thickness and a truncation to produce a required order of approximation. Since Legendre polynomials are not appropriate eigenfunctions of the branches of thickness vibrations, the obtained two-dimensional theory cannot describe cut-off frequencies and long-wave asymptotes of those branches. The "correction coefficients" are introduced to improve the match between the frequency spectra of an infinite plate as obtained from the approximate and exact equations.

Although Mindlin's and Mindlin-Medick's theories have been successfully applied for many engineering problems (see Mindlin, the Collected Papers, and quotations therein), their introduction of the "correction coefficients" remains a little bit mystic. Berdichevsky (1977) was the first who showed that the long-wave asymptotic analysis can be applied for branches of high-frequency thickness vibrations of elastic plates near the cut-off frequencies. Based on the variational asymptotic method (Berdichevsky, 1979, 1983) he found the distributions of the displacements and derived the equations of high frequency long-wave vibrations for all thickness branches. This method is then applied for elastic rods (Kvashnina, 1979), elastic shells (Berdichevsky and Le, 1980), and piezoelectric plates (Le, 1984). The later checking, by Kaplunov (1990), confirm the results for plates, but display some arithmetic mistakes in our calculations of coefficients for the equations of shells, the corrections of which lead to the full agreement of the results.

The equations derived in Berdichevsky and Le (1980) are asymptotically exact and describe correctly the behavior of shells in the long-wave range near the cut-off frequencies. However, these same equations without modification yields an unsatisfactory description of the dispersion curves and the group velocities in the short-wave range. At the same time, the formulation of boundary-value problems is associated with the behavior of the differential operator at short wavelength. Thus, even asymptotic exact equations in the long-wave range may lead to the ill-posed boundary value problems (Berdichevsky, 1979). Therefore the construction of the theory of shells involves not only the derivation of equations in the long-wave range, but also another logically independent step—the extrapolation of those equations to short waves.

It is possible to carry out either trivial extrapolations, when the system of equations derived for long waves is applied for short waves without any changes, or nontrivial extrapolations, when terms that are small in the long-wave range but appreciable for short waves are introduced (removed). We clarify this by a simple example. Let us consider the following one-dimensional equations

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \tag{1}$$

$$\partial_t^2 u - c^2 \partial_x^2 u + c^2 h^2 \partial_x^4 u = 0. \tag{2}$$

Here c is a constant having the dimension of velocity, h is a small parameter having the dimension of length. In the long-wave range these equations are indistinguishable in the first approximation, for the term $c^2 h^2 \partial_x^4 u$ for long waves is small compared with the term $c^2 \partial_x^2 u$. In the short-wave range, however, eqns (1) and (2) differ essentially. Equation (1) is a hyperbolic equation of second order requiring two boundary conditions. In eqn (2), for short waves, the term $c^2 \partial_x^2 u$, which is small compared with $c^2 \partial_x^4 u$, can be neglected, so that (2) is analogous to the equation for the transverse vibrations of a beam and requires the formulation of four boundary conditions. Equations (1) and (2) can therefore be regarded as the two different short-wave extrapolations describing the same physical situation in the long-wave range.

For shells in the short-wave range it is impossible to describe the three dimensional stress state exactly by the two-dimensional theory, and only a qualitative agreement can at best be expected. For this reason, different two-dimensional equations are allowed in the theory of shells. However, it is natural to demand asymptotic equivalence in the long-wave range of the different short-wave extrapolations.

In our paper (Berdichevsky and Le, 1982) the best hyperbolic short-wave extrapolation is proposed for the equations derived in Berdichevsky and Le (1980). This involves the classical branches and several thickness branches of vibrations and takes into account their cross-terms at short waves. The structure of the equations is similar to those of Mindlin and Mindlin and Medick for plates, but in contrast to their theories, the asymptotic accuracy is achieved in the long-wave range by the asymptotic analysis and not by an introduction of "correction coefficient" based on an ad hoc assumption. This result is then generalized for piezoelectric shells (Le, 1985, 1986), and elastic rods (Le, 1986).

In this paper we present the full derivation of the two-dimensional theory of high-frequency vibration of the elastic shells. We regard this theory as the shell theory with internal degrees of freedom, which is specified by the two-dimensional kinetic and internal energy densities. We then apply this theory to study wave propagation in an infinite cylindrical shell. We compare the dispersion curves according to the two- and three-dimensional theory (Gazis, 1959) and show the asymptotical exactness of the former in the long-wave range. Based on the two-dimensional theory, an edge mode of vibration is found for the semi-infinite cylindrical shells. The physical explanation of the existence of this mode is given.

2. DISPLACEMENT NEAR THE CUT-OFF FREQUENCIES

The long-wave state in shells is defined as the state whose smallest wavelength l of the deformation pattern in the *longitudinal* directions is considerably greater than the shell thickness h. The possible types of long-wave vibrations can be classified roughly as follows. Let the face surfaces of the shell be traction-free. Since $l \gg h$, the derivatives of the displacements with respect to the longitudinal coordinates ξ^α ($\alpha = 1, 2$) can be neglected in the Lame equations and in the stress free boundary conditions as small compared with the derivatives with respect to the transverse coordinates $\xi \in (-h/2, h/2)$. Then the Lame equations decompose into a system of three independent equations

$$\mu \frac{\partial^2 w_\alpha}{\partial \xi_3{}^2} = \rho \ddot{w}_\alpha, \quad |\xi| \leqslant \frac{h}{2}, \quad \frac{\partial w_\alpha}{\partial \xi} = 0, \quad \xi = \pm \frac{h}{2}, \tag{3}$$

$$(\lambda + 2\mu) \frac{\partial^2 w}{\partial \xi^2} = \rho \ddot{w}, \quad |\xi| \leqslant \frac{h}{2}, \quad \frac{\partial w}{\partial \xi} = 0, \quad \xi = \pm \frac{h}{2}. \tag{4}$$

Here w_α and w are projections of the displacements on the longitudinal and transverse coordinates, λ and μ are Lame's constants of the isotropic elastic material of the shell, and ρ the mass density. Small Greek indices correspond to projections on the longitudinal coordinates and range over 1, 2. The dot over quantities denotes their time derivative. The complete set of particular solutions of (3) and (4) follows

$$w = v\sqrt{2} \cos \alpha\zeta, \quad w_\alpha = 0, \quad \alpha = 2\pi n, \quad (F_\perp(n)), \tag{5}$$

$$w = 0, \quad w_\alpha = \psi_\alpha \sqrt{2} \sin \beta\zeta, \quad \beta = \pi(2n+1), \quad (F_\parallel(n)), \tag{6}$$

$$w = \psi\sqrt{2} \sin \alpha\zeta, \quad w_\alpha = 0, \quad \alpha = \pi(2n+1), \quad (L_\perp(n)), \tag{7}$$

$$w = 0, \quad w_\alpha = v_\alpha \sqrt{2} \cos \beta\zeta, \quad (L_\parallel(n)), \tag{8}$$

with $\zeta = \xi/h$. The quantities α and β run through a countable set of values, however, no indices are attached to α and β in order to avoid complicated notations. The factor $\sqrt{2}$ is chosen so as to simplify the two-dimensional kinetic energy. It is understood that the functions v, ψ_α, ψ, and v_α correspond to each value of α or β; these functions are also not numbered. Furthermore, they depend harmonically on t with frequency ω which is determined by the appropriate values of α or β from the formulae

$$\omega = \frac{\alpha c_1}{h} \quad \text{or} \quad \omega = \frac{\beta c_2}{h}. \tag{9}$$

Here c_1 and c_2 are the velocities of dilatational and shear waves, respectively. The notation for series of different solutions is indicated in parentheses in (5)–(8).

For functions, v, ψ_α, ψ, v_α independent of ξ^α, each of the solutions given above represents an exact solution of the Lame equations for an infinite plate and corresponds to synchronized vibrations of transverse fibers along the plate (with the zero longitudinal wave number). The frequencies (9) will be called cut-off frequencies. For vibrations whose amplitude and frequency vary slowly in the longitudinal directions of the plates and shells, the equations (3)–(4) can be regarded as the zero approximations. The solutions (5)–(8) can be considered as the principal terms in a certain asymptotic expansion in which v, ψ_α, ψ, v_α are functions of ξ^α and t, where

$$\dot{v} \sim \omega u, \quad \dot{\psi}_\alpha \sim \omega\psi_\alpha, \quad \dot{\psi} \sim \omega\psi, \quad \dot{v}_\alpha \sim \omega u_\alpha. \tag{10}$$

The values of ω in these estimates are taken for the same branch as the corresponding function, with the exception of $F_\perp(0)$ and $L_\|(0)$, for which it is assumed that $\dot{v} \sim c_1 v/l$, $\dot{v}\alpha \sim c_2 v_\alpha/l$, where l is the smallest wavelength of the deformation pattern. The branches $F_\perp(0)$ and $L_\|(0)$ correspond to the low frequency vibration when $\omega h/c_1 \ll 1$. The independence of the displacement at these branches from ξ in the zero approximation is a part of the Kirchhoff-Love hypothesis (Kirchhoff, 1850, Love, 1927). All the remaining branches correspond to vibrations with frequency $\omega \sim c_1/h$. The propagation time for a perturbation over the thickness is of the same order as the period of vibration, and it is impossible to suppose the displacements polynomials in ξ even in a zero approximation. Since $\omega \to \infty$ as $h \to 0$, the corresponding vibrations are naturally called high-frequency (or thickness) vibrations.

Taking for example $n = 1$, $c_2 = 2500$ m/s (e.g., steel), $h = 3$ mm, we have $\omega_1 \simeq 4 \cdot 10^5$ Hz for the branch $F_\|(0)$, i.e., ω_1 is in the ultrasonic domain. Vibrations of elastic bodies at such a frequency can be important in problems of impact or in problems of vibrations caused by an electromagnetic field (Mason, 1950, Shaw, 1956). Let us note that for layered shells of sandwich type with a significant drop in the elastic moduli, ω_1 is considerably smaller and can even be in the audio frequency domain (Ryazantseva, 1985). The branches have the displacement distributions which oscillate all the more rapidly over ξ as n grows. The distribution of the branch n has $2n$ or $2n+1$ zeros. Note that the wavelength of the high-frequency branches in the transverse direction of the three-dimensional shell is smaller than h, but this fact is not an obstacle for the application of the asymptotic analysis, which is based on the smallness of h/l, with l the wavelength in the longitudinal direction.

To formulate the problem of free vibration of the shell we refer its unreformed state to the curvilinear coordinates ξ^α, ξ

$$x^i = r^i(\xi^\alpha) + \xi n^i(\xi^\alpha).$$

Here, x^i are Cartesian coordinates, $r^i(\xi^\alpha)$ is the position vector of the middle surface Ω, $n^i(\xi^\alpha)$ is the normal to Ω. Latin indices correspond to projections on x^i and range over 1, 2, 3. In the coordinate system ξ^α, ξ the covariant and contravariant components of the metric tensors are given by the formulas

$$\begin{aligned} g_{\alpha\beta} &= a_{\alpha\beta} - 2b_{\alpha\beta}\xi + c_{\alpha\beta}\xi^2, \quad g_{\alpha 3} = 0, \quad g_{33} = 1, \\ g^{\alpha\beta} &= \frac{1}{\kappa^2}[(1-2H\xi)a^{\alpha\beta} + 2\xi(1-2H\xi)b^{\alpha\beta} + \xi^2 c^{\alpha\beta}] \\ &= a^{\alpha\beta} + 2b^{\alpha\beta}\xi + 3c^{\alpha\beta}\xi^2 + O\left(\frac{h^3}{R^3}\right), \\ g^{\alpha 3} &= 0, \quad g^{33} = 1. \end{aligned} \tag{11}$$

Here $a_{\alpha\beta}$, $b_{\alpha\beta}$, $c_{\alpha\beta}$ are the first, second and third quadratic forms of the middle surface Ω, respectively, $\kappa = 1 - 2H\xi + K\xi^2$, H and K are the mean and Gaussian curvatures of Ω, R is the smallest radius of curvature of Ω. We express the components of the strain tensor in the following form

$$\varepsilon_{\alpha\beta} = x^i_{,(\alpha} w_{iL,\beta)} = r^i_{(\alpha} w_{i,\beta)} - \xi b^\sigma_{(\alpha} r^i_\sigma w_{i,\beta)}$$

$$= w_{(\alpha;\beta)} - w b_{\alpha\beta} - \xi b^\sigma_{(\alpha} w_{\sigma;\beta)} + \xi w c_{\alpha\beta},$$

$$2\varepsilon_{3\alpha} = n^i w_{i,\alpha} + x^i_{,\alpha} w_{i,\xi} = w_{,\alpha} + b^\sigma_\alpha w_\sigma + w_{\alpha,\xi} - b^\sigma_\alpha \xi w_{\sigma,\xi},$$

$$\varepsilon_{33} = n^i w_{i,\xi} = w_{,\xi}. \tag{12}$$

Here $w_\alpha = w_i r^i_\alpha$, $w = w_i n^i$ are the projections of the displacement vector w_i on the tangent vectors $r^i_\alpha = r^i_{,\alpha}$ and the normal n^i, the comma in the subscripts denotes partial differentiation with respect to ξ^α, the semicolon denotes covariant differentiation on the surface Ω, and the parentheses in the subscripts denote the symmetrization operation.

According to Hamilton's principle the displacements corresponding to the free vibrations of the shell are extremals of the action functional (see, e.g., Berdichevsky, 1983)

$$I = \int_{t_1}^{t_2} \int_\Omega \int_{-h/2}^{h/2} L\kappa \, d\xi \, da \, dt, \tag{13}$$

with L the Lagrangian given by

$$L = \frac{1}{2}\rho(\dot{w}^2 + a^{\alpha\beta}\dot{w}_\alpha \dot{w}_\beta) - \left[\frac{1}{2}\lambda(g^{\alpha\beta}\varepsilon_{\alpha\beta} + \varepsilon_{33})^2 + \mu g^{\alpha\gamma} g^{\beta\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \mu(\varepsilon_{33})^2 + 2\mu g^{\alpha\beta}\varepsilon_{\alpha 3}\varepsilon_{33}\right], \tag{14}$$

and da the area element on Ω.

Let us apply the variational-asymptotic method to the variational problem of finding the extremals of the functional (13). We assume the smallness of the parameters $h_* = h/R$ and $h_{**} = h/l$ everywhere in Ω. Making the substitution $\zeta = \xi/h$ and discarding formally all small terms in (13) we arrive at the zero approximation functional

$$I_0 = h\int_{t_1}^{t_2} \int_\Omega \int_{-1/2}^{1/2} \left[\frac{1}{2}\rho\dot{w}^2 + \frac{1}{2}\rho a^{\alpha\beta}\dot{w}_\alpha \dot{w}_\beta - (\lambda + 2\mu)\frac{1}{2h^2}(w_{,\zeta})^2 - \mu\frac{1}{2h^2} a^{\alpha\beta} w_{\alpha,\zeta} w_{\beta,\zeta}\right] d\zeta \, da \, dt. \tag{15}$$

The Euler equations of this functional yield four series of free vibrations (5)–(8). It is convenient to introduce further in the series $F_\perp$ and $L_\perp$ the number β according to

$$\alpha = e\beta, \quad e = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1-2\nu}{2-2\nu}}. \tag{16}$$

Similarly, we also introduce in the series $F_\parallel$ and $L_\parallel$ the number α by the same formula (16).

Now we find the next refinement for the displacements of branches of the series $F_\perp$. Considering v belonging to the branch $F_\perp(n)$ a given function of ξ^α and t, we seek w_α for that branch. Keeping the principal terms depending on w_α and the principal cross terms in (13), we obtain the functional

$$I_1 = h\int_{t_1}^{t_2} \int_\Omega \int_{-1/2}^{1/2} L_1 \, d\zeta \, da \, dt, \tag{17}$$

with

$$L_1 = \frac{1}{2}\rho\omega^2 w_\alpha w^\alpha - \lambda w^\alpha v_{,\alpha}\alpha h^{-1}\sqrt{2}\sin\alpha\zeta$$
$$-\frac{1}{2}\mu\alpha^{\alpha\beta}(h^{-1}w_{\alpha,\zeta}+v_{,\alpha}\sqrt{2}\cos\alpha\zeta)(h^{-1}w_{\beta,\zeta}+v_{,\beta}\sqrt{2}\cos\alpha\zeta).$$

Integration by parts was performed and terms that go to the boundary and does not affect the equations inside Ω were neglected in (17). Let us find the extremal of this functional. After taking the variation of (17) relative to w_α we obtain the equations

$$w_{\alpha,\zeta\zeta}+\beta^2 w_\alpha = \frac{\lambda+\mu}{\mu}h\alpha v_{,\alpha}\sqrt{2}\sin\alpha\xi, \quad |\xi| \leqslant 1/2, \quad w_{,\alpha}+hv_{,\alpha}\sqrt{2}\cos\alpha\zeta = 0, \quad \zeta = \pm 1/2. \tag{18}$$

They yield the following tangential displacements

$$w_\alpha = v_{,\alpha}\frac{h}{\alpha}\sqrt{2}\left(\sin\alpha\zeta - \frac{2(-1)^n e\sin(\beta\zeta)}{\cos(\beta/2)}\right). \tag{19}$$

As expected, the tangential displacements turn out to be much smaller than the normal displacement in the long-wave range and are of the order $h_{**}u$.

Let us seek the correction to w

$$w = v\cos\alpha\zeta + w'.$$

Here w_α is considered fixed and defined by (19). Without limiting the generality, the following constraint can be imposed on w'

$$\int_{-1/2}^{1/2} w'\cos\alpha\zeta\,\mathrm{d}\zeta = 0.$$

It corresponds to the assumption that $u = \langle w\sqrt{2}\cos\alpha\zeta\rangle$, where $\langle . \rangle$ denotes the integration over ζ from $-1/2$ to $1/2$. After discarding small terms containing w' and small cross terms as compared with the rest, the functional (13) takes the form (17) with a Lagrangian given by the formula

$$L_1 = \frac{1}{2}\rho\dot{w}^2 + \rho 2Hh\zeta\dot{v}\sqrt{2}\cos\alpha\zeta\dot{w}' - \frac{1}{2h^2}(\lambda+2\mu)(w'_{,\zeta})^2$$
$$-\frac{(\lambda+2\mu)}{h}2H\zeta\alpha v\sqrt{2}\sin\alpha\zeta w'_{,\zeta} + \lambda\frac{2H}{h}w'_{,\zeta}v\sqrt{2}\cos\alpha\zeta - \lambda\frac{2H}{h}w'\alpha v\sqrt{2}\sin\alpha\zeta. \tag{20}$$

Its extremal has the form

$$w' = Hhv\sqrt{2}(\zeta\cos\alpha\zeta + \frac{1-4e^2}{\alpha}\sin\alpha\zeta).$$

Summing up, we have the following distribution of the displacements over the thickness in the series $F_\perp$ (within the first approximation)

$$F_{\perp}:\quad w = v\sqrt{2}\cos\alpha\zeta + Hhv\sqrt{2}\left(\zeta\cos\alpha\zeta + \frac{1-4e^2}{\alpha}\sin\alpha\zeta\right),$$

$$w_\alpha = v_{,\alpha}\frac{h}{\alpha}\sqrt{2}\left(\sin\alpha\zeta - \frac{2(-1)^n e\sin(\beta\zeta)}{\cos(\beta/2)}\right). \tag{21}$$

Analogously, formulae are obtained for the displacements in the three remaining series

$$F_{\parallel}:\quad w_a = \psi_\alpha\sqrt{2}\sin\beta\zeta + h\sqrt{2}\left(H\psi_\alpha\zeta\sin\beta\zeta + \frac{\bar{b}^\beta_\alpha}{\beta}\psi_\beta\cos\beta\zeta\right),$$

$$w = \psi^\alpha_{;\alpha}\frac{h}{\beta}\sqrt{2}\left(\cos\beta\zeta - \frac{2(-1)^n e\cos(\alpha\zeta)}{\sin(\alpha/2)}\right), \tag{22}$$

$$L_{\perp}:\quad w = \psi\sqrt{2}\sin\alpha\zeta + Hh\psi\sqrt{2}\left(\zeta\cos\alpha\zeta - \frac{1-4e^2}{\alpha}\cos\alpha\zeta\right),$$

$$w_\alpha = \psi_{,\alpha}\frac{h}{\alpha}\sqrt{2}\left(-\cos\alpha\zeta + \frac{2(-1)^n e\cos(\beta\zeta)}{\sin(\beta/2)}\right), \tag{23}$$

$$L_{\parallel}:\quad w_\alpha = v_\alpha\sqrt{2}\cos\beta\zeta + h\sqrt{2}\left(Hv_\alpha\zeta\cos\beta\zeta - \frac{\bar{b}^\beta_\alpha}{\beta}v_\beta\sin\beta\zeta\right),$$

$$w = v^\alpha_{;\alpha}\frac{h}{\beta}\sqrt{2}\left(-\sin\beta\zeta + \frac{2(-1)^n e\sin(\alpha\zeta)}{\cos(\alpha/2)}\right), \tag{24}$$

where $\bar{b}^\beta_\alpha = b^\beta_\alpha + H\delta^\beta_\alpha$. The distinguishing feature of shells as compared with plates is that the correction terms in the displacements are of the order h_* compared to the principal term, while they are of the order $h^2{}_{**}$ in plates. By continuing the iteration process, the next corrections to w and w_α can be found. They are not presented here since they yield no contribution to the average Lagrangian of the first approximation.

3. THICKNESS VIBRATIONS IN THE LONG-WAVE RANGE

Let the displacements w, w_α be expressed by the infinite series of branches given above, where u, ψ_α, ψ, u_α are arbitrary functions of ξ^α and t. After substituting these series into the action functional (13) and integrating over the thickness we neglect those small terms of order h_*, h_{**} compared with 1. It turns out that the thickness branches are orthogonal relative to the energy functional in the long-wave range (Berdichevsky, 1977, Berdichevsky and Le, 1980). Therefore the average functional has the form

$$I = h\int_{t_1}^{t_2}\int_\Omega \bar{L}\,\mathrm{d}a\,\mathrm{d}t, \tag{25}$$

where the average Lagrangian $\bar{L}$ decomposes into a series of average Lagrangians of low frequency and thickness branches.† For the series $F_\perp$ we get

† The average Lagrangian of the low frequency branches can be found in Berdichevsky, 1979.

$$F_{\perp}:\quad 2\bar{L} = \rho\dot{v}^2+\rho l_2\left(\frac{h}{\alpha}\right)^2 a^{\alpha\beta}\dot{v}_{,\alpha}\dot{v}_{,\beta}+\rho l_4\left(\frac{h}{\alpha}\right)^2\dot{v}^2$$

$$-(\lambda+2\mu)\left(\left(\frac{\alpha}{h}\right)^2 v^2+l_1\alpha^{\alpha\beta}v_{,\alpha}v_{,\beta}+l_3v^2\right),$$

$$l_1 = 2\left(1-\frac{2e^2\tan(\beta/2)}{\beta/2}\frac{5-3e^2}{1-e^2}+\frac{2e^2}{\cos^2(\beta/2)}\right),$$

$$l_2 = 1-\frac{3-e^2}{1-e^2}\frac{4e^2\tan(\beta/2)}{\beta/2}+\frac{4e^2}{\cos^2(\beta/2)},$$

$$l_3 = -(3H^2-K)\left(\frac{3}{2}+\frac{\alpha^2}{12}-8e^2\right)+4H^2(1-5e^2+4e^4),$$

$$l_4 = -(3H^2-K)\left(\frac{1}{2}+\frac{\alpha^2}{12}\right)+2H^2(1-6e^2+8e^4). \tag{26}$$

Within the first approximation one can further simplify this expression. Indeed, at the cut-up frequencies eqns (9) and (10) are valid so that one can replace the term $\rho l_2(h/\alpha)^2a^{\alpha\beta}\dot{v}_{,\alpha}\dot{v}_{,\beta}$ by $\rho l_2 c_1^2 a^{\alpha\beta}v_{,\alpha}v_{,\beta}$ and the term $\rho l_4(h/\alpha)^2\dot{v}^2$ by $\rho l_4 c_1^2 v^2$. Now the average Lagrangians for the series $F_{\perp}$ become

$$F_{\perp}:\quad 2\bar{L} = \rho\dot{v}^2-\mu[(h^{-2}\beta^2+k_2)v^2+k_1a^{\alpha\beta}v_{,\alpha}v_{,\beta}],$$

$$k_1 = \frac{1}{e^2}-\frac{16\tan(\beta/2)}{\beta},$$

$$k_2 = -H^2\left(\frac{1}{e^2}-16\right)+K\left(\frac{1}{e^2}-8\right). \tag{27}$$

In the similar way we get

$$F_{\parallel}:\quad 2\bar{L} = \rho a^{\alpha\beta}\dot{\psi}_{\alpha}\dot{\psi}_{\beta}-\mu[(h^{-2}\beta^2a^{\alpha\beta}+k_2^{\alpha\beta})\psi_{\alpha}\psi_{\beta}+2\psi_{(\alpha;\beta)}\psi^{(\alpha;\beta)}+k_1(\psi^{\alpha}_{;\alpha})^2],$$

$$k_1 = -1+\frac{16e^2\cot(\alpha/2)}{\alpha}, \tag{28}$$

$$k_2^{\alpha\beta} = (3H^2-K)a^{\alpha\beta}+6Hb^{\alpha\beta}-2Ka^{\alpha\beta},$$

$$L_{\perp}:\quad 2\bar{L} = \rho\dot{\psi}^2-\mu[(h^{-2}\beta^2+k_2)\psi^2+k_1a^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta}],$$

$$k_1 = \frac{1}{e^2}+\frac{16\cot(\beta/2)}{\beta}, \tag{29}$$

$$L_{\parallel}:\quad 2\bar{L} = \rho a^{\alpha\beta}\dot{v}_{\alpha}\dot{v}_{\beta}-\mu[(h^{-2}\beta^2a^{\alpha\beta}+k_2^{\alpha\beta})v_{\alpha}v_{\beta}+2v_{(\alpha;\beta)}v^{(\alpha;\beta)}+k_1(v^{\alpha}_{;\alpha})^2],$$

$$k_1 = -1-\frac{16e^2\tan(\alpha/2)}{\alpha}. \tag{30}$$

The coefficients k_2 in the series $L_{\perp}$ and the tensors $k_2^{\alpha\beta}$ in the series $L_{\parallel}$ are not written down here since they agree in form with those in the series $F_{\perp}$ and $L_{\parallel}$.† Not only the principal terms containing the factor 1 in the kinetic energy and the factor h^{-2} in the internal energy,

† There were some arithmetic mistakes in our calculation of the coefficients of correction terms in Berdichevsky and Le (1980).

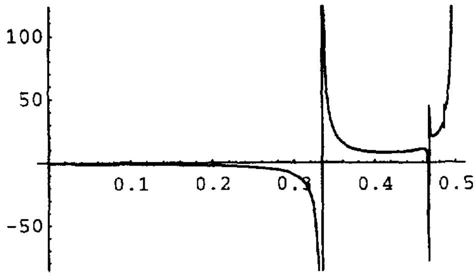

Fig. 1. Graph of k_1 as the function of v.

but also terms of the next orders of smallness must be retained in the average Lagrangians (27)–(30), due to the fact that, at the cut-up frequencies, the sum of the principal terms turns out to be small.

By varying the action functional (25) with $\bar{L}$ from (27)–(30) we arrive at the following equations of thickness vibration

$$F_\perp : \quad \rho\ddot{v} = \mu[-(h^{-2}\beta^2 + k_2)v + k_1 \Delta v], \tag{31}$$

$$F_\parallel : \quad \rho\ddot{\psi}_\alpha = \mu[-(h^{-2}\beta^2\psi_\alpha + k^\beta_{2\alpha}\psi_\beta) + (k_1+1)\psi^\beta_{;\beta\alpha} + \Delta\psi_\alpha]. \tag{32}$$

The equations for the series $L_\perp$ and $L_\parallel$ can be obtained from (31) and (32) by making the respective substitutions: $v \to \psi$, $\psi_\alpha \to v_\alpha$. These equations coincide with those of Kaplunov (1990) derived by the asymptotic method of Goldenveizer.

It is interesting to note that the type of the eqns (31)–(32) depend on the coefficients k_1. For instance, eqn (31) is of the hyperbolic type if $k_1 > 0$. Figure 1 shows the graphs of k_1 as the functions of the Poisson ratio v for the branch $L_\perp(0)$. One can see that for the range $0 < v < 1/3$ this coefficient is negative, and the equation of vibration is of the elliptic type.

We shall see in the next section, how this "pathological" feature of the equations of thickness vibration could be removed by extrapolating them to the short waves taking into account the cross terms between branches.

4. HYPERBOLIC SHORT-WAVE EXTRAPOLATION

Let us consider vibrations of the shell that can be regarded with sufficient accuracy as the superposition of the branches $F_\perp(0)$, $F_\parallel(0)$, $L_\parallel(0)$, $L_\perp(0)$, $L_\parallel(1)$. The branches $F_\perp(0)$ and $L_\parallel(0)$ correspond to low-frequency vibrations, the other ones—to thickness vibrations with the lowest frequencies. The dynamic equations contain eight unknown functions of the longitudinal coordinates and the time: $\bar{u}$, $\bar{u}^\alpha$, $\bar{\psi}^\alpha$, $\bar{\psi}$, $\bar{v}^\alpha$ (the symbols without the bar are reserved for the functions in the final equations). Despite the fact that the theory involves more unknown functions than in the classical shell theory, it should be regarded as a first approximation theory describing asymptotically exactly the vibrations of the shell in the range of long waves and high frequencies ($\omega \leqslant 2\pi c_2/h$).

Thus, we present the displacements of the shell in the form

$$\begin{aligned} w &= \bar{u} - h\sigma\bar{A}^\nu_\nu\zeta + h^2\bar{\rho}^\nu_\nu m(\zeta) + \bar{\psi}a(\zeta) + h\bar{v}^\lambda_{;\lambda}g(\zeta), \\ w_\alpha &= \bar{u}_\alpha - hn^i\bar{u}_{i,\alpha}\zeta + h^3\bar{\rho}^\lambda_{\alpha;\lambda}p(\zeta) + h^3\bar{\rho}^\lambda_{\lambda;\alpha}q(\zeta) \\ &\quad + \bar{\psi}_\beta e^\beta_\alpha(\zeta) + \bar{v}_\alpha d^\beta_\alpha(\zeta) + h\bar{\psi}_{,\alpha} f(\zeta) \end{aligned} \tag{33}$$

with $\sigma = \lambda/(\lambda+2\mu) = v/(1-v)$. The three first terms of the first equations of (33) and four of the second one describe the low-frequency branches (Berdichevsky, 1979), with the following surface strain measures

$$\bar{A}_{\alpha\beta} = n^i_{(\alpha}\bar{u}_{i,\beta)} = \bar{u}_{(\alpha;\beta)} - b_{\alpha\beta}\bar{u},$$
$$\bar{\rho}_{\alpha\beta} = (n^i_{i,(\alpha})_{;\beta)} + b^{\lambda}_{(\alpha}\bar{\omega}_{\beta)\lambda},$$
$$\bar{\omega}_{\alpha\beta} = \frac{1}{2}(\bar{u}_{\beta,\alpha} - \bar{u}_{\alpha,\beta}). \tag{34}$$

The functions $m(\zeta)$, $p(\zeta)$, $q(\zeta)$, $a(\zeta)$, $d^{\beta}_{\alpha}(\zeta)$, $e^{\beta}_{\alpha}(\zeta)$, $f(\zeta)$, $g(\zeta)$ are given by

$$m(\zeta) = \frac{1}{2}(\zeta^2 - 1/12), \quad p(\zeta) = \frac{1}{3}\left(\zeta^3 - \frac{3}{4}\zeta\right), \quad q(\zeta) = \frac{1}{6}\left(\zeta^3 - \frac{5}{4}\zeta\right),$$
$$a(\zeta) = \sqrt{2}\sin\pi\zeta + Hh\sqrt{2}(\zeta\cos\pi\zeta - \frac{1-4e^2}{\pi}\cos\pi\zeta),$$
$$d^{\beta}_{\alpha}(\zeta) = \sqrt{2}\delta^{\beta}_{\alpha}\cos 2\pi\zeta + h\sqrt{2}(H\delta^{\beta}_{\alpha}\zeta\cos 2\pi\zeta - \frac{\bar{b}^{\beta}_{\alpha}}{2\pi}\sin 2\pi\zeta),$$
$$e^{\beta}_{\alpha}(\zeta) = c\delta^{\beta}_{\alpha}\sin\pi\zeta + hc(H\delta^{\beta}_{\alpha}\zeta\sin\pi\zeta + \frac{\bar{b}^{\beta}_{\alpha}}{\pi}\sin\pi\zeta),$$
$$f(\zeta) = \frac{\sqrt{2}}{\pi}\left(-\cos\pi\zeta + \frac{2e\cos(\pi\zeta/e)}{\sin(\pi/2e)}\right),$$
$$g(\zeta) = \frac{\sqrt{2}}{2\pi}\left(-\sin 2\pi\zeta - \frac{2e\sin(2\pi e\zeta)}{\cos(\pi e)}\right), \tag{35}$$

with c at the moment an undefined constant that will be chosen later to simplify the subsequent change of unknown functions. In (33) we neglect the correction term associated with $h\bar{\psi}^{\lambda}_{;\lambda}$. This is due to an additional analysis, which shows that a hyperbolic short-wave extrapolation describing exactly the curvature of the dispersion curve near the cut-up frequency of the branch $F_{\|}(0)$ does not exist.

We substitute the formulae (33) into the action functional (13) and integrate over the thickness. Discarding small terms in the asymptotic sense and using the results of the previous section, after long but otherwise standard calculations one can show that

$$\begin{aligned}\bar{L} = \frac{1}{2}\rho\Big[&\dot{\bar{u}}^2 + \frac{c^2}{2}(\dot{\bar{\psi}}_{\alpha})^2 - (\dot{\bar{u}}_{\alpha})^2 + 2c_1 h\dot{\bar{u}}_{\alpha}\dot{\bar{\psi}}^{;\alpha} + \dot{\bar{\psi}}^2 + 2c_2 h\dot{\bar{\psi}}\dot{\bar{A}}^{\lambda}_{\lambda}\\
&+ 2c_3 h\dot{\bar{\psi}}\dot{\bar{v}}^{\lambda}_{;\lambda} + \dot{\bar{v}}^2_{\alpha} + 2c_4 h\dot{\bar{v}}_{\alpha}\dot{\bar{\psi}}^{;\alpha}\Big] - \frac{1}{2}\mu\Big[\beta_2^2 h^{-2}\bar{\psi}^2 + \beta_3^2 h^{-2}\bar{v}^2_{\alpha} + 2d_2 h^{-1}\bar{\psi}\bar{v}^{\lambda}_{;\lambda}\\
&+ 2d_3 h^{-1}\bar{v}^{\alpha}\bar{\psi}_{,\alpha} + 2\sigma(\bar{A}^{\lambda}_{\lambda})^2 + 2\bar{A}^{\alpha\beta}\bar{A}_{\alpha b} + k_2\bar{\psi}^2_{,\alpha} + k_3(\bar{v}^{\lambda}_{;\lambda})^2 + 2\bar{v}^{(\alpha;\beta)}\bar{v}_{(\alpha;\beta)}\\
&+ \frac{h^2}{6}(\sigma(\bar{\rho}^{\lambda}_{\lambda})^2 + \bar{\rho}^{\alpha\beta}\bar{\rho}_{\alpha\beta}) + 2hd_4\bar{\rho}^{\lambda}_{\lambda}\bar{\psi}^{\lambda}_{;\lambda} + 2hd_5\bar{\rho}^{\alpha\beta}\bar{\psi}_{(\alpha;\beta)} + h^{-2}\beta_1^2\frac{c^2}{2}\bar{\psi}^2_{\alpha}\\
&+ 2hd_6\bar{\psi}^{\alpha}\bar{\rho}^{\lambda}_{\lambda;\alpha} + 2hd_7\bar{\psi}^{\alpha}\bar{\rho}^{\lambda}_{\alpha;\lambda} + (c^2/2)s_4^{\alpha\beta}\bar{\psi}_{\alpha}\bar{\psi}_{\beta} + s_5\bar{\psi}^2 - s_6^{\alpha\beta}\bar{v}_{\alpha}\bar{v}_{\beta}\Big].\end{aligned} \tag{36}$$

In this Lagrangian the coefficients are given by the following formulae

$$\beta_1 = \pi, \quad \beta_2 = \frac{\pi}{e}, \quad \beta_3 = 2\pi,$$
$$k_2 = \frac{1}{e^2} + \frac{16e}{\pi}\cot\left(\frac{\pi}{2e}\right), \quad k_3 = -1 - \frac{8e}{\pi}\tan(\pi e),$$

$$c_1 = c_2 = -\frac{2\sqrt{2}\sigma}{\pi^2}, \quad c_3 = c_4 = \frac{1}{\pi^2}\left(-\frac{4}{3}+\frac{2e^2}{e^2-1/4}\right),$$

$$d_2 = d_3 = \beta_2^2 c_3 + r_2, \quad r_2 = \frac{4\sigma}{3e^2}, \quad r_3 = -\frac{16}{3},$$

$$d_4 = d_6 = \frac{4\sigma c}{\pi^2}, \quad d_5 = d_7 = \frac{4c}{\pi^2},$$

$$s_4^{\alpha\beta} = (3H^2-K)a^{\alpha\beta}+6Hb^{\alpha\beta}-2Ka^{\alpha\beta},$$

$$s_5 = -H^2(1/e^2-16)+K(1/e^2-8). \tag{37}$$

It is interesting to note the identifies $c_1 = c_2$, $c_3 = c_4$, $d_2 = d_3$, $d_4 = d_6$, $d_5 = d_7$, which mean that the cross-terms in (36) form divergence terms that do not affect the equations of vibrations in the long-wave range.

In order to search for a short-wave extrapolation which does not contain second and higher derivatives in the Lagrangian let us choose $c = \pi^2/24$† and make the following substitutions

$$u = \bar{u}, \quad \psi_\alpha = -n^i\bar{u}_{i,\alpha}+h^{-1}\left(\bar{\psi}_\alpha-\frac{\sigma}{3}\left(\frac{24}{\pi^3}\right)^2\bar{\rho}^\lambda_{\lambda;\alpha}-\frac{1}{3}\left(\frac{24}{\pi^3}\right)^2\bar{\rho}^\lambda_{\alpha;\lambda}\right),$$

$$u_\alpha = \bar{u}_\alpha + c_1 h\bar{\psi}_{,\alpha}, \quad \psi = \bar{\psi}+c_1 h\bar{A}^\lambda_\lambda + c_3 h\bar{v}^\lambda_{;\lambda}, \quad v_\alpha = \bar{v}_\alpha + c_3 h\bar{\psi}_{,\alpha}. \tag{38}$$

The sense of these changes of unknown functions is to make all terms containing second and higher derivatives of the new functions negligibly small in the long-wave range. Discarding them and extrapolating the result to short wave, we obtain the following average Lagrangian.

$$\begin{aligned}\bar{L} = \frac{1}{2}\rho(\dot{u}^2+\dot{u}_\alpha^2+\alpha h^2\dot{\psi}_\alpha^2+\dot{\psi}^2+\dot{v}_\alpha^2)-\frac{\mu}{2}\Big[& s_1(A^\lambda_\lambda)^2+2A^{\alpha\beta}A_{\alpha\beta}\\ &+\frac{h^2}{6}(\sigma(\rho^\lambda_\lambda)^2+\rho^{\alpha\beta}\rho_{\alpha\beta})+s_2\psi^2_{,\alpha}+s_3(v^\lambda_{;\lambda})^2+2v^{(\alpha;\beta)}b_{(\alpha;\beta)}\\ &+2r_1h^{-1}\psi A^\lambda_\lambda+2r_2h^{-1}\psi v^\lambda_{;\lambda}+2r_3h^{-1}v^\alpha\psi_{,\alpha}\\ &+f_1^{\alpha\beta}(\psi_\alpha+n^iu_{i,\alpha})(\psi_\beta+n^iu_{i,\beta})+f_2h^{-2}\psi^2+f_3^{\alpha\beta}h^{-2}v_\alpha v_\beta\Big],\end{aligned} \tag{39}$$

with

$$A_{\alpha\beta} = u_{(\alpha;\beta)}-b_{\alpha\beta}u, \quad \rho_{\alpha\beta} = -\psi_{(\alpha;\beta)}+b^\lambda_{(\alpha}\omega_{\beta)\lambda}, \quad \omega_{\alpha\beta} = \frac{1}{2}(u_{\beta,\alpha}-u_{\alpha,\beta}). \tag{40}$$

The new coefficients in (39) are given by

$$\alpha = \frac{1}{2}\left(\frac{\pi^2}{24}\right)^2, \quad f_1^{\alpha\beta} = \alpha(\pi^2a^{\alpha\beta}+h^2s_4^{\alpha\beta}),$$

$$f_2 = \left(\frac{\pi}{e}\right)^2+h^2s_5, \quad f_3^{\alpha\beta} = (2\pi)^2a^{\alpha\beta}+h^2s_4^{\alpha\beta},$$

† It is easy to show that the Lagrangian does not depend on this special choice.

$$r_1 = \frac{2\sqrt{2}\sigma}{e^2}, \quad s_1 = 2\sigma + \frac{8\sigma^2}{e^2\pi^2},$$

$$s_2 = \frac{1}{e^2} + \frac{16e\cot(\pi/2e)}{\pi} + \frac{16(1+2e^2)^2}{9\pi^2(4e^2-1)e^2} - \frac{8\sigma^2}{\pi^2 e^2},$$

$$s_3 = -1 - \frac{8e\tan(\pi e)}{\pi} - \frac{16(1+2e^2)^2}{9\pi^2(4e^2-1)e^2}. \tag{41}$$

The Lagrangian $\bar{L}$ is a quadratic form with respect to u_α, u, ψ_α, ψ, v_α and their first derivatives. Varying the action functional with L from (39), we obtain the equations

$$\begin{aligned}
\rho h\ddot{u}^\alpha &= t^{\alpha\beta}_{;\beta} - q^\beta b^\alpha_\beta, \\
\rho h\ddot{u} &= q^\alpha_{;\alpha} + t^{\alpha\beta} b_{\alpha\beta}, \\
\rho\alpha h^3 \ddot{\psi}^a &= -q^\alpha - m^{\alpha\beta}_{;\beta}, \\
\rho h\ddot{\psi} &= \mu h(s_2\Delta\psi - r_1 h^{-1} A^\lambda_\lambda - r_{23} h^{-1} v^\lambda_\lambda - f_2 h^{-2}\psi), \\
\rho h\ddot{v}_\alpha &= \mu h((s_3+1)v^\lambda_{;\lambda\alpha} + \Delta v_\alpha + r_{23} h^{-1}\psi_{,\alpha} - f^\beta_{3\alpha} h^{-2} v_\beta),
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
t^{\alpha\beta} &= n^{\alpha\beta} + \frac{1}{2}(b^\alpha_\lambda m^{\lambda\beta} - b^\beta_\lambda m^{\lambda\alpha}), \\
n^{\alpha\beta} &= \mu h(s_1 A^\lambda_\lambda a^{\alpha\beta} + 2A^{\alpha\beta} + r_1 h^{-1}\psi a^{\alpha\beta}), \\
m^{\alpha\beta} &= \mu\frac{h^3}{6}(\sigma\rho^\lambda_\lambda a^{\alpha\beta} + \rho^{\alpha\beta}), \quad q^\alpha = \mu h f^{\alpha\beta}_1(\psi_\beta + u_{,\beta} + b^\lambda_\beta u_\lambda),
\end{aligned} \tag{43}$$

and

$$r_{23} = r_2 - r_3.$$

It is easy to show that when $s_2 > 0$ and $s_3+1 > 0$ then the internal energy (the expression in the square brackets of (39)) is positive definite and the Euler equations (42) are of the hyperbolic type. Figure 2 depicts the graphs of s_2 and s_3+1 as functions of v in the interval $(0, 0.44)$, which are positive over there.

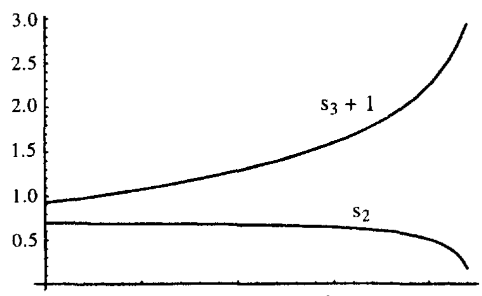

Fig. 2. Graphs of s_2 and s_3+1 as functions of v.

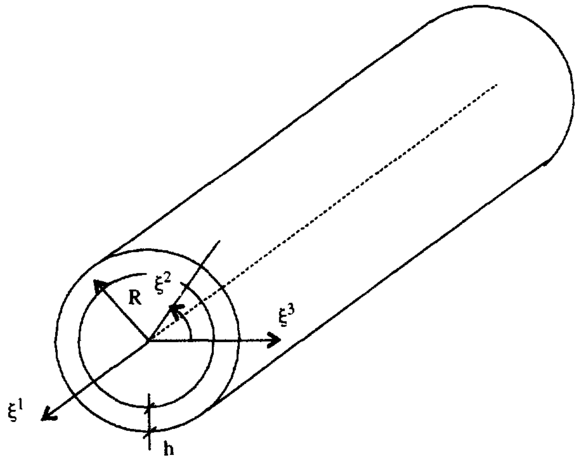

Fig. 3. A cylindrical shell.

5. WAVE PROPAGATION AND AN EDGE MODE IN CYLINDRICAL SHELLS

Let us consider a cylindrical coordinate system ξ^1, ξ^2, ξ^3, with ξ^1 the axial, $\xi^2 = R\theta$ the circumferential, and ξ^3 the radial coordinate, respectively. An infinitely long cylindrical shell (Fig. 3) occupies the region $|\xi^2| \leqslant \pi R$, $|\xi^3 - R| \leqslant h/2$. Its middle surface is specified by $\xi^3 = R$, which leads to the following first and second quadratic forms

$$a_{11} = a_{22} = 1, \quad a_{12} = 0; \quad b_{11} = b_{12} = 0, \quad b_{22} = -\frac{1}{R}.$$

We seek solutions of the wave equations (42) for the cylindrical shell in the form

$$(u_1, u, \psi_1, \psi, v_1) = (w_1, w_3, w_4, w_6, w_7)\cos\eta\xi^2 e^{i(k\xi^1 - \omega t)},$$
$$(u_2, \psi_2, v_2) = (w_2, w_5, w_8)\sin\eta\xi^2 e^{i(k\xi^1 - \omega t)}, \quad (44)$$

where the unknowns $w_1, \ldots, w_8$ do not depend on ξ^α and t, and η is determined by

$$\eta R = n, \quad n = 0, 1, 2, \ldots$$

Substituting (44) into (42) and eliminating the common factors, which is either $\cos\eta\xi^2 \exp[i(k\xi^1 - \omega t)]$ or $\sin\eta\xi^2 \exp[i(k\xi^1 - \omega t)]$, we arrive at the following eigenvalue problem

$$\mathbf{H}(k)\mathbf{w} = \omega^2\mathbf{w}, \quad (45)$$

where $\mathbf{H}$ is an 8×8 matrix, whose elements are the (complex) functions of k. One can check directly that for real k there are eight real eigenvalues of (45). In Fig. 4 graphs of the dimensionless frequencies $\vartheta = \omega h/(\pi c_2)$ vs the dimensionless wave numbers $\kappa = kh/(2\pi)$ (dispesion curves) are shown. The parameters chosen for the numerical calculation are equal to

$$\nu = 0.3, \quad h/R = 1/30, \quad n = 1.$$

To be able to compare with the analogous results from the three dimensional theory, we show also the dispersion curves obtained numerically by Gazis (1958) (the dashed lines in Fig. 4). One can see that both curves are almost identical in the long-wave range. Moreover, even in the short-wave range there is a qualitatively good agreement between them. In Fig. 5 we present a detail of the lowest three modes near the origin, where the dispersion curves according to the two- and three-dimensional theories are practically identical.

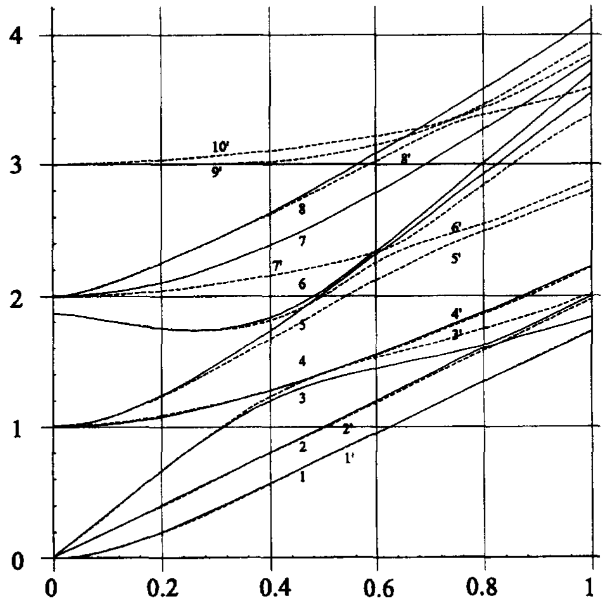

Fig. 4. Dispersion curves for $n = 1$, $h/R = 1/30$.

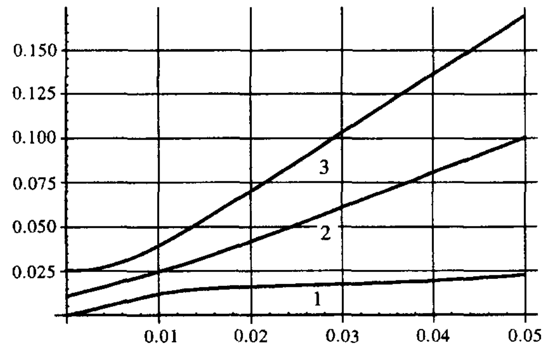

Fig. 5. Detail of the three lowest branches for $n = 1$, $h/R = 1/30$.

One of the most remarkable features of the thickness vibrations of semi-infinite shells, plates and rods is the existence of a so-called edge mode (Mindlin and Medick, 1959). Let us consider the semi-infinite cylindrical shell bounded by a free edge at $\xi^1 = 0$ and assume that

$$u_2 = \psi_2 = v_2 = 0, \quad u_1, u, \psi_1, \psi, v_1 \text{ do not depend on } \xi^2.$$

This corresponds to the case $n = 0$. If, additionally, the ratio h/R is small, then the interaction between F- and L-branches can be considered as negligible and consequently the equations for u_1, ψ, v_1 and those for u, ψ_1 are uncoupled. Introducing the dimensionless coordinate $\zeta^1 = \xi^1/h$ and time $\tau = tC_2/h$, the former can be rewritten as follows

$$\begin{aligned} u_{1|\tau\tau} &= S_1 u_{1|11} + r_1 \psi_{|1}, \\ \psi_{|\tau\tau} &= S_2 \psi_{|11} - r_1 u_{1|1} - r_{23} v_{1|1} - f_2 \psi, \\ v_{1|\tau\tau} &= S_3 v_{1|11} + r_{23} \psi_{|1} - f_3 v_1, \end{aligned} \tag{46}$$

where the vertical bar preceding indices denotes the partial derivative with respect to the corresponding dimensionless coordinate, with the following coefficients

$$S_1 = s_1+2, \quad S_2 = s_2, \quad S_3 = s_3+1,$$
$$f_3 = (2\pi)^2+3(h/R)^2/4.$$

Seeking the solution of (46) in the form

$$(u_1, \psi, v_1) = (\tilde{u}_1, \tilde{\psi}_1, \tilde{v}_1)e^{i\vartheta\tau}$$

we reduce (46) to

$$S_1\tilde{u}_{1|11}-r_1\tilde{\psi}_{|1}+\vartheta^2\tilde{u}_1 = 0,$$
$$S_2\tilde{\psi}_{1|11}-r_1\tilde{u}_{1|1}-r_{23}\tilde{v}_{1|1}+(\vartheta^2-f_2)\tilde{\psi} = 0,$$
$$S_3\tilde{v}_{1|11}+r_{23}\tilde{\psi}_{|1}+(\vartheta^2-f_3)\tilde{v}_1 = 0. \tag{47}$$

The functions $\tilde{u}_1$, $\tilde{\psi}_1$, $\tilde{v}_1$ can be expressed in terms of three potential function Φ_i, $i = 1, 2, 3$ according to

$$\tilde{u} = \Phi_{1|1}+\Phi_{2|1}+\Phi_{3|1},$$
$$\tilde{\psi}_1 = a_1\Phi_1+a_2\Phi_2+a_3\Phi_3,$$
$$\tilde{v}_1 = b_1\Phi_{1|1}+b_2\Phi_{2|1}+b_3\Phi_{3|1}. \tag{48}$$

It is easy to show that $\tilde{u}_1$, $\tilde{\psi}_1$, $\tilde{v}_1$ are the solution of (46) if and only if

$$\Phi_{i|11}+\kappa_i^2\Phi_i = 0, \quad i = 1,2,3, \tag{49}$$

where κ_i^2 are the three roots of the dispersion equation

$$(S_1\kappa^2-\vartheta^2)(S_2\kappa^2-\vartheta^2+f_2)(S_3\kappa^2-\vartheta^2+f_3)-(S_3\kappa^2-\vartheta^2+f_3)r_1^2\kappa^2-(S_1\kappa^2-\vartheta^2)r_{23}^2\kappa^2 = 0, \tag{50}$$

and where

$$a_i = (S_1\kappa_i^2-\vartheta^2)/r_1,$$
$$b_i = r_{23}a_i/(S_3\kappa_i^2-\vartheta^2+f_3).$$

Let us consider the interval of frequencies, in which only one real root κ_1^2 of the cubic equation (50) exists. The other two roots $\kappa_{2,3}^2$ are complex conjugate to each other. We present the solution of (49) in the form

$$\Phi_1 = e^{i\kappa_1\zeta^1}+A_1e^{-i\kappa_1\zeta^1},$$
$$\Phi_2 = A_2e^{-i\kappa_2\zeta^1}, \quad \Phi_3 = A_3e^{-i\kappa_3\zeta^1}. \tag{51}$$

Because the solution should remain bounded as $\zeta^1 \to \infty$, κ_2 and κ_3 should be taken in the lower half of the complex plane. Substituting (48) and (51) into the free boundary conditions

$$S_1\tilde{u}_{1|1}+r_1\tilde{\psi} = 0,$$
$$S_2\tilde{\psi}_{|1}+r_3\tilde{v} = 0,$$
$$S_3\tilde{v}_{1|1}+r_2\tilde{\psi} = 0, \tag{52}$$

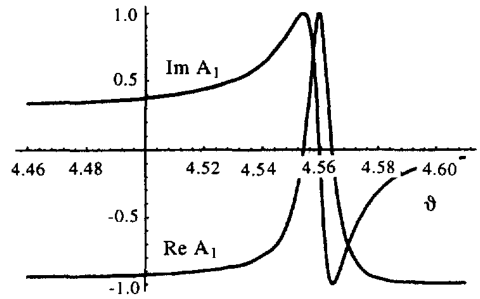

Fig. 6. The real and imaginary parts of A_1.

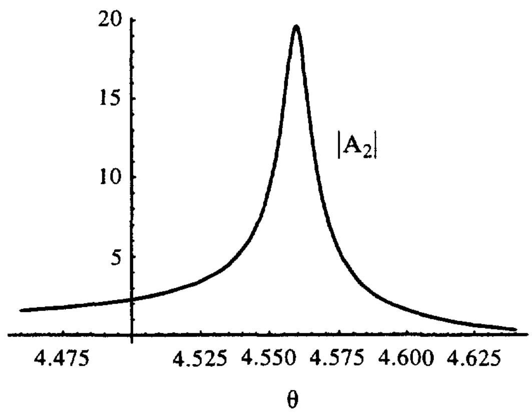

Fig. 7. Graph of $|A_2|$.

at $\zeta^1 = 0$, we obtain the system of three linear equations with respect to A_i. In Fig. 6 graphs of $\mathrm{Re}(A_1)$ and $\mathrm{Im}(A_1)$ as functions of ϑ are shown (Poisson's ratio ν is equal to 0.31). At $\vartheta \approx 4.56$ we have $\mathrm{Re}(A_1) = 1$ and $\mathrm{Im}(A_1) = 0$. This corresponds to the frequency of the edge mode. In Fig. 7 one can see the graph of $|A_2|$ as a function of ϑ. At this frequency $|A_2|$ reaches it maximum.

6. CONCLUDING REMARKS

In this paper the following results are obtained:

1. The improved 2D equations of high frequency vibrations of elastic shells are derived by using the variational-asymptotic method, which guarantees their accuracy in the long-wave range.
2. The improved hyperbolic short-wave extrapolation of the 2D equations of high frequency vibrations in the range of frequencies $0 \leqslant \omega \leqslant 2\pi c_2/h$ is provided. The derived equations are asymptotically exact in the long-wave range and have a qualitatively good agreement with the 3D theory in the short-wave range.
3. The dispersion curves according to the derived 2D equations of high frequency vibrations are calculated numerically for infinitely long cylindrical shells. Comparison of 2D and 3D dispersion curves confirms the accuracy of the 2D equations. The edge mode in the semi-infinite cylindrical shell is studied in detail, the explanation of its existence is provided.

Acknowledgements—The author is grateful to Prof. V. L. Berdichevsky (Detroit) for his encouragement in writing this paper.

REFERENCES

Berdichevsky, V. L. (1977) High-frequency long-wave vibrations of plates. *Dokl. Akad. Nauk SSSR* **236**, 1319–1322.

Berdichevsky, V. L. (1979) Variational asymptotic method of constructing the theory of shells. *Applied Mathematics and Mechanics (PMM)* **43,** 664–687.

Berdichevsky, V. L. (1983) *Variational principles of continuum mechanics.* Nauka, Moscow (in Russian).

Berdichevsky, V. L., Le, K. C. (1980) High-frequency long-wave shell vibration. *Applied Mathematics and Mechanics (PMM)* **44,** 737–744.

Berdichevsky, V. L. and Le, K. C. (1982) High-frequency vibrations of shells. *Dokl. Akad. Nauk SSSR* **267,** 584–589.

Gazis, D. C. (1959) Three-dimensional investigation of the propagation of waves in hollow cylinders, P.I. Analytical foundation; P.II Numerical results. *Journal of Acoustical Society of America* **31,** 627–634.

Kaplunov, J. D. (1990) High-frequency stress-strain states. *Izv. Akad. Nauk SSSR, Mekh. Tverdogo Tela* **25,** 147–157.

Kirchhoff, G. (1850) Ueber das Gleichgewicht und die Bewegung einer elastischen Scheibe. *Journal reine und angew. Mathematics* **40,** 51–88.

Kvashnina, S. S. (1979) High-frequency long-wave vibrations of elastic rods. *Applied Mathematics and Mechanics (PMM)* **43,** 335–341.

Lamb, H. (1917) On waves in an elastic plate. *Proceedings of the Royal Society, London* **A93,** 114–128.

Le, K. C. (1982) High-frequency long-wave vibrations of piezoelectric ceramic plates. *Dokl. Akad. Nauk SSSR* **264,** 581–583.

Le, K. C. (1985) High-frequency vibrations of piezoelectric ceramic shells *Dokl. Akad. Nauk SSSR* **284,** 1327–1330.

Le, K. C. (1986) The theory of piezoelectric shells. *Applied Mathematics and Mechanics (PMM)* **50,** 136–146.

Le, K. C. (1986) High-frequency longitudinal vibrations of elastic rods. *Applied Mathematics and Mechanics (PMM)* **50,** 443–450.

Love, A. E. H. (1927) *A treatise on the Mathematical Theory of Elasticity.* Cambridge Uni. Press, Cambridge.

Mason, W. P. (1950) *Piezoelectric crystals and their applications to Ultrasonics.* D. Van Nostrand Company, New York.

Mindlin, R. D. (1951) Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *Journal of Applied Mechanics* **18,** 31–38.

Mindlin, R. D. and Medick, M. A. (1959) Extensional vibrations of elastic plates. *Journal of Applied Mechanics* **26,** 561–569.

Mindlin, R. D. (1989) *The Collected Papers.* Springer, Berlin.

Pochhammer, L. (1876) Über die Fortplanzungsgeschwindigkeiten kleiner Schwingungen in einem unbegrenzten isotropen Kreiszylinder. *Zeitschrift für Mathematik* **81,** 324–336.

Lord Rayleigh (1889) On the free vibrations of an infinite plate of homogeneous isotropic elastic matter. *Proceedings of the Royal Mathematical Society, London* **10,** 225–234.

Reissner, E. (1945) The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics, Trans. ASME* **67,** A-69.

Ryazantseva, M. Yu. (1985) Flexural vibrations of symmetrical sandwich plates. *Mechanics of Solids (MTT)* **20,** 153–159.

Shaw, E. A. G. (1956) On the resonant vibrations of thick barium titanate disks. *Journal of Acoustical Society of America* **28,** 38–50.

Timoshenko, S. (1922) On the transverse vibrations of bars of uniform cross section. *Philosophical Magazine* **43,** 125–131.